

Non-perturbatively Renormalized Light-Quark Masses with the Alpha Action

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Abstract:

We have computed the light quark masses using the $\mathcal{O}(a^2)$ improved Alpha action, in the quenched approximation. The renormalized masses have been obtained non-perturbatively. By eliminating the systematic error coming from the truncation of the perturbative series, our procedure removes the discrepancies, observed in previous calculations, between the results obtained using the vector and the axial-vector Ward identities. It also gives values of the quark masses larger than those obtained by computing the renormalization constants using (boosted) perturbation theory. Our main results, in the RI (MOM) scheme and at a renormalization scale $\mu = 2$ GeV, are $m_s^{RI} = 138(15)$ MeV and $m_l^{RI} = 5.6(5)$ MeV, where m_s^{RI} is the mass of the strange quark and $m_l^{RI} = (m_u^{RI} + m_d^{RI})/2$ the average mass of the up-down quarks. From these results, which have been obtained non-perturbatively, by using continuum perturbation theory we derive the \overline{MS} masses, at the same scale, and the renormalization group invariant (m^{RGI}) masses. We find $m_s^{NLO \overline{MS}} = 121(13)$ MeV and $m_l^{NLO \overline{MS}} = 4.9(4)$ MeV at the next-to-leading order; $m_s^{N^2LO \overline{MS}} = 111(12)$ MeV, $m_l^{N^2LO \overline{MS}} = 4.5(4)$ MeV, $m_s^{RGI} = 177(19)$ MeV and $m_l^{RGI} = 7.2(6)$ MeV at the next-to-next-to-leading order.

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1 Introduction

The values of the quark masses obtained from lattice simulations have recently attracted the attention in the physics community. The reason is that, for these quantities, the lattice method is unique: from the non-perturbative calculation of hadronic quantities, this approach allows in fact a consistent determination of the quark masses, defined as effective couplings renormalized at short distances. Yet, with the errors quoted by the authors [1]–[8], the values of the light and strange quark masses computed in different numerical simulations (mainly in the quenched approximation) are often in disagreement. The differences originate mainly from two sources: on the one hand, from the different procedures used to compute the renormalized mass from the bare lattice one; on the other, from the different methods used to extrapolate results, obtained at finite values of a , to the continuum limit.

Our numerical calculations have been done with the non-perturbatively improved fermion action (which we will denote as the Alpha action) introduced in ref. [9], see also [10]–[15]. The use of a non-perturbatively improved action and operators reduces discretization errors to $\mathcal{O}(a^2)$. Thus, at least for light quarks (namely the u , d and s quarks), we expect these errors to be rather small, i.e. much smaller than other systematic effects. The other source of uncertainty arises from the truncation of the perturbative series in the definition of the renormalized mass. This problem can be eliminated by using a non-perturbative method for renormalizing the lattice operators. In our study, we have computed the relevant renormalization constants Z_S and Z_P , in the chiral limit, using the non-perturbative renormalization procedure on quark states proposed in ref. [16] (in the following we denote this method as NPM). As a check of the accuracy of the NPM, we have also computed the renormalization constants of the vector and axial-vector currents and compared the results with those obtained in refs. [14], see also [10]. At $\beta = 6.0$ and 6.2 , our results for these currents agree within less than 5% with previous determinations. This makes us confident that the values of Z_S and Z_P determined with the NPM are correct. Note that, at the same values of β , the value of Z_P (Z_S) computed in boosted perturbation theory is larger than our estimate by about 30% (by about 10%)¹. This implies that the use of perturbation theory leads to an underestimate of the quark mass by an amount much larger than the expected discretization errors². It also makes the two determinations of the masses, from the vector and axial vector Ward identities, different. With the NPM, instead, we find essentially the same value of the quark masses with the two methods.

This letter is focused on the calculation of the light quark masses: we describe the procedure followed in the determination of the bare masses, \tilde{m} , discuss the definition of the renormalized and of the renormalization group invariant (RGI) ones, \hat{m} and m^{RGI} , and present the final results and errors for these quantities. All details of the (standard) analysis of the light hadron spectrum and decay constants, together with a study of the

¹ We used for the boosted coupling [17, 18] $g^2 = g_0^2/\langle P \rangle$, where the average plaquette at $\beta = 6.2$ ($\beta = 6.0$) is given $\langle P \rangle = 0.6136$ ($\langle P \rangle = 0.5937$) as inferred by our simulations.

² The observation that the NPM gives larger quark masses was first made in ref. [1], and then confirmed in a systematic study at different values of β and with different actions in ref. [7].

energy-momentum relation, can be found in ref. [19]. For the light hadron spectrum, the calibration of the lattice spacing, the values of the Wilson parameter in the chiral limit, κ_{crit} , and the values of κ corresponding to the light (up and down) and strange quark masses, we substantially agree with previous studies of the same quantities [6, 8]. We performed two independent simulations: a test run at $\beta = 6.0$ on a $16^3 \times 32$ lattice (with a sample of 45 configurations at four values of the light quark masses) and a run at $\beta = 6.2$ on a $24^3 \times 64$ lattice (on a sample of 100 configurations at four values of the light quark masses). The final results are based on the “data” at $\beta = 6.2$. Those of the test run have only been used as a consistency check of the stability of the physical values of the quark masses. We will compare the results at $\beta = 6.0$ with those at $\beta = 6.2$ at the end of the paper.

2 Definition of the quark masses

We now explain the procedure followed to extract the physical values of the quark masses. The starting point are the vector and axial-vector Ward identities (WI) [20, 21]

$$\nabla_\mu \langle \alpha | \hat{V}_\mu | \beta \rangle = (\hat{m}_1(\mu) - \hat{m}_2(\mu)) \langle \alpha | \hat{S}(\mu) | \beta \rangle, \quad (2.1)$$

$$\nabla_\mu \langle \alpha | \hat{A}_\mu | \beta \rangle = (\hat{m}_1(\mu) + \hat{m}_2(\mu)) \langle \alpha | \hat{P}(\mu) | \beta \rangle, \quad (2.2)$$

where $|\alpha\rangle$ and $|\beta\rangle$ represent generic external physical states; \hat{V}_μ and \hat{A}_μ are the normalized currents which obey the current algebra commutation relations; $\hat{S}(\mu)$, $\hat{P}(\mu)$ and $\hat{m}_i(\mu)$ are operators and quark masses renormalized at the scale μ in a given scheme. Note that the products $\hat{m}(\mu)\hat{S}(\mu)$ and $\hat{m}(\mu)\hat{P}(\mu)$ are regularization, renormalization and scale independent.

Ward identities as (2.1) and (2.2) can also be written for the lattice currents. In this case they are valid up to terms of $\mathcal{O}(a)$, $\mathcal{O}(\alpha_s a)$ or $\mathcal{O}(a^2)$, depending whether we use the Wilson, the tree-level improved [22] or non-perturbatively improved actions and operators³.

In order to determine the renormalized masses it is obvious that we have first to specify the scheme used to renormalize the scalar and pseudoscalar densities. Given the uncertainties of lattice perturbation theory, we have renormalized these quantities by imposing, non-perturbatively, the following renormalization conditions [16]

$$\begin{aligned} \langle q(p) | \hat{S}(\mu) | q(p) \rangle &\equiv Z_S(\mu) \langle q(p) | S | q(p) \rangle = 1, \\ \langle q(p) | \hat{P}(\mu) | q(p) \rangle &\equiv Z_P(\mu) \langle q(p) | P | q(p) \rangle = 1, \end{aligned} \quad (2.3)$$

in the chiral limit. In the above equations $|q(p)\rangle$ represents an external off-shell quark state with virtuality $p^2 = \mu^2$, and the matrix elements are evaluated in the Landau gauge. The conditions (2.3) at large values of μ^2 ensure that \hat{S} and \hat{P} belong to the same chiral multiplet, i.e. they obey to the relevant Ward identities. It can be shown that this is true

³ On the lattice ∇_μ is written in terms of (improved) finite differences.

also at the improved level, i.e. that, using the improved action of ref. [14], the matrix elements of \hat{S} and \hat{P} renormalized as in eq. (2.3) have discretization errors of $\mathcal{O}(a^2)$ [23].

Since vector symmetries are preserved by the lattice regularization, unlike axial vector ones, the procedures which are usually employed to extract the quark masses using the vector or axial vector Ward identities are different and we discuss them separately.

- **Determination of quark masses from the vector Ward identity.**

From the lattice version of eq. (2.1), one finds

$$\hat{m}(\mu) = Z_S^{-1}(\mu, ma, 0) \frac{1}{2a} \left(\frac{1}{\kappa} - \frac{1}{\kappa_{crit}} \right), \quad (2.4)$$

where, in the definition of Z_S , we added those terms which are necessary to improve the scalar density out of the chiral limit

$$Z_S(\mu, m_1 a, m_2 a) = Z_S(\mu) \left(1 + \frac{b_S}{2} (m_1 a + m_2 a) \right). \quad (2.5)$$

By defining $Z_S(\mu) = Z_m^{-1}(\mu)$ and $b_S = -2b_m$, from eq. (2.4) we derive the standard relation

$$\hat{m}(\mu) \equiv Z_m(\mu) \tilde{m} = Z_m(\mu) \left[\frac{1}{2a} \left(\frac{1}{\kappa} - \frac{1}{\kappa_{crit}} \right) \left(1 + b_m \frac{1}{2a} \left(\frac{1}{\kappa} - \frac{1}{\kappa_{crit}} \right) \right) \right]. \quad (2.6)$$

We have used the above equation to determine the quark mass, with $Z_m(\mu) = Z_S(\mu)^{-1}$ as computed from eq. (2.3), κ_{crit} as fixed from the squared pseudoscalar meson mass, the calibration of the lattice spacing either from m_ρ or from m_{K^*} and $1/\kappa = 1/\kappa_{u,d,s}$ from the meson spectroscopy. The discussion of the values of Z_S and Z_P and the relative errors can be found in sec. 3.

As far as b_m is concerned, the method of ref. [23] has difficulties in computing it. The gauge invariant procedure of ref. [24] has not been applied to date. A different approach to fix b_m non-perturbatively has been proposed in ref. [25]. We have tried the same technique and found that the results for b_m (and for all the analogous quantities such as b_A , b_V etc.) are very unstable and that these constants cannot be determined reliably [26]⁴. For this reason, we preferred to take b_m from boosted perturbation theory, using $b_m = b_m^{(0)} + b_m^{(1)} g^2 = -1/2 - 0.0962 g^2 = -0.652$ [28]. If the uncertainty on the perturbative value of b_m were as large as $b_m^{(1)} g^2$, this would induce a relative error of $\mathcal{O}(6 \cdot 10^{-4})$ and $\mathcal{O}(4 \cdot 10^{-3})$ on $(\tilde{m}_d + \tilde{m}_u)/2$ and \tilde{m}_s , respectively. Given the other sources of errors (due to the uncertainty in the determination of Z_S and to the quenched approximation), this error is negligible and will be ignored in the following.

In ref. [2], they call HS (from Hadron Spectrum) the method based on eq. (2.6). This is rather misleading because the possibility of relating $1/2a(1/\kappa - 1/\kappa_{crit})$, or any

⁴A new method for a non-perturbative determination of the b s, based on the lattice chiral Ward identities, has been recently presented at the Lattice '98 conference [27]. However, this method has not yet been implemented with the non-perturbatively improved Alpha action.

improved version of it, to the quark mass, entirely relies on the validity of the lattice vector WI. The hadron spectrum enters in this case, as in the case of the axial-vector WI, only because it is needed to fix the bare quark mass from the physical value of some hadronic quantity.

- **Determination of quark masses from the axial-vector Ward identity.**

In this case, the simplest procedure is to use the axial Ward identity computed on hadron states at rest, for degenerate quark masses. One gets

$$\begin{aligned}\hat{m}(\mu) &\equiv \frac{Z_A}{Z_P(\mu)} \tilde{\rho} = \frac{Z_A(ma, ma)}{Z_P(\mu, ma, ma)} \frac{\langle \alpha | \nabla_0 A_0 + ac_A \nabla_0^2 P | \beta \rangle}{2a \langle \alpha | P | \beta \rangle} \\ &= \frac{Z_A}{Z_P(\mu)} \left[\frac{1 + b_A ma}{1 + b_P ma} \frac{\langle \alpha | \nabla_0 A_0 + ac_A \nabla_0^2 P | \beta \rangle}{2a \langle \alpha | P | \beta \rangle} \right],\end{aligned}\tag{2.7}$$

where b_A and b_P play the same role for the axial current and the pseudoscalar density as b_S in the case of the scalar operator. In the absence of a precise non-perturbative determination of these quantities, we have used, as we did above for b_m , boosted perturbation theory, namely $b_A = 1 + 0.1522g^2 = 1.240$ and $b_P = 1 + 0.1531g^2 = 1.241$ [28]. The same considerations made for b_m about the systematic error induced by perturbation theory in the calculation of b_A and b_P on the masses of the light quarks, remain true in this case. We extracted the quark mass by computing the ratio in eq. (2.7) at values of κ corresponding to the physical meson masses (m_π , m_K and m_ϕ), taking the non-perturbative determinations of Z_A and $Z_P(\mu)$, evaluated in the chiral limit.

3 Non-perturbatively renormalized quark masses

In this section, we first discuss the values and errors of the renormalization constants $Z_S(\mu)$, $Z_P(\mu)$ and Z_A . We then explain the procedure followed to extract the bare masses, \tilde{m} and $\tilde{\rho}$, using eqs. (2.6) and (2.7). From the Z s and from the values of the bare masses, we have obtained the values of the non-perturbatively renormalized quark masses in the RI scheme. Using perturbation theory, we then get the \overline{MS} and RGI masses at the NLO and N^2LO .

The relevant Green functions (vertices and propagators) necessary to the determination of the Z s with the NPM have been computed with external quark states of virtuality μ on the same configurations, and for the same values of κ , as all the other quantities discussed in this paper. We used quark propagators improved following the strategy of refs. [23, 26]. The Green functions have then been extrapolated (linearly in the quark masses) to the chiral limit.

As explained in ref. [16], the NPM is expected to work when μ satisfies the condition $\Lambda_{QCD} \ll \mu \ll 1/a$. In this region, Z_A and Z_P/Z_S should be independent of μ , since the non-perturbative method is equivalent to the Ward identities. We monitored the behaviour

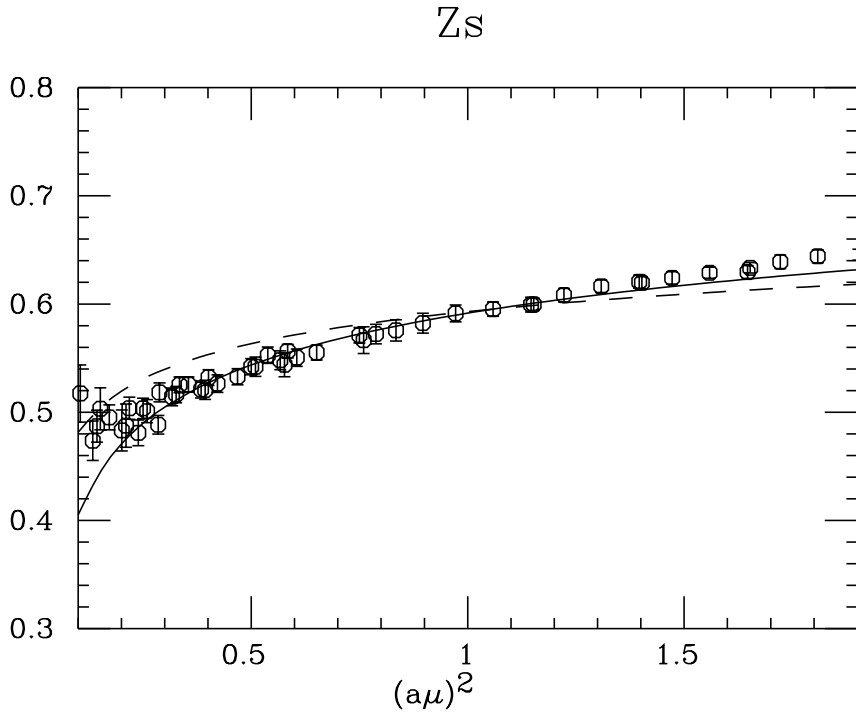


Figure 1: Z_S obtained by using the NPM as a function of the scale $\mu^2 a^2$. The dashed and solid curves represent the solutions of the renormalization group equations at the leading and next-to-leading order, respectively.

of Z_A and Z_P/Z_S as a function of μ to find the range of μ where these quantities exhibit a plateau. At $\beta = 6.2$, we find that Z_A is essentially a constant for $\mu^2 a^2 \leq 2$, while Z_P/Z_S has a plateau for $1 \leq \mu^2 a^2 \leq 2$, corresponding to $2.8 \text{ GeV} \leq \mu \leq 4.0 \text{ GeV}$. From the analysis of the plateaus of Z_A and Z_P/Z_S , we have obtained $Z_A = 0.793(5)$, in agreement with the value $Z_A = 0.809$ of ref. [14], and $Z_P/Z_S = 0.78(1)$. The latter is in disagreement with the boosted result $Z_P/Z_S = 0.95$. Similar results, although with larger statistical fluctuations, and systematic uncertainties, have been found at $\beta = 6.0$ [26].

As a consistency check of our results, we have also studied the μ dependence of Z_S . In fig. 1 we show the behaviour of Z_S as a function of μ compared with the solution of the renormalization group equations

$$Z_S(\mu) = Z_S(\mu_0) \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right)^{\gamma_0/2\beta_0} \left(1 + \frac{\alpha_s(\mu) - \alpha_s(\mu_0)}{4\pi} J \right), \quad (3.8)$$

where, in Landau RI and with $n_f = 0$ [29],

$$\gamma_0 = -8, \quad \gamma_1 = -252, \quad \beta_0 = 11, \quad \beta_1 = 102$$

$$J = \frac{\gamma_1 \beta_0 - \gamma_0 \beta_1}{2\beta_0^2}. \quad (3.9)$$

By fitting the numerical results to eq. (3.8), with Λ_{QCD} as free parameter, we get $\Lambda_{QCD}^{n_f=0} = 200 \pm 55 \text{ MeV}$ in good agreement with the determination of ref. [31], $\Lambda_{QCD}^{n_f=0} \sim 251 \pm 21 \text{ MeV}$. It is reassuring that eq. (3.8) fits rather well the behaviour of $Z_S(\mu)$ down to rather low values of μ^2 . From the fit, and using $Z_P/Z_S = 0.78(1)$, we find the results given in

$\beta = 6.2$				
μ	Z_S (NPM)	Z_S (BPT)	Z_P (NPM)	Z_P (BPT)
2.80 GeV	0.60(1)	0.66	0.47(1)	0.62
2.00 GeV	0.55(1)	0.61	0.43(1)	0.57
$\beta = 6.0$				
μ	Z_S (NPM)	Z_S (BPT)	Z_P (NPM)	Z_P (BPT)
2.00 GeV	0.55(3)	0.63	0.39(3)	0.59

Table 1: Z_S and Z_P obtained by using the NPM and boosted perturbation theory (BPT), at two typical reference scales, $\mu = 2.80$ GeV, corresponding to $\mu^2 a^2 = 1$ at $\beta = 6.2$, and $\mu = 2$ GeV which is the reference renormalization scale for the quark masses in lattice calculations. We also used the values of Z_A as obtained with the NPM, namely $Z_A = 0.793(5)$ and $Z_A = 0.78(1)$ at $\beta = 6.2$ and 6.0 respectively.

table 1. We stress that the use of perturbation theory induces an error of 10–30% in the extraction of the values of the quark masses, depending whether one uses the vector or axial-vector Ward identity (i.e. Z_S or Z_P). For completeness we also give the results at $\beta = 6.0$. More details on the analysis of the Z s obtained with the NPM can be found in ref. [26] ⁵.

Eqs. (2.6) and (2.7) determine the bare masses \tilde{m} and $\tilde{\rho}$ as a function of the hopping parameter κ . The physical values of κ , i.e. κ_l , κ_s etc., are fixed, together with the lattice spacing, using a certain number of physical conditions. We now explain the procedure followed in our analysis.

We fit the mass of the vector meson to the expression

$$aM_V = C + L(aM_{PS})^2 + Q(aM_{PS})^4, \quad (3.10)$$

where M_{PS} is the pseudoscalar mass. The cases with $Q \neq 0$ or $Q = 0$ are denoted as quadratic or linear fit respectively. By taking $M_{PS} = m_\pi$ ($M_{PS} = m_K$) and $M_V = m_\rho$ ($M_V = m_{K^*}$), where m_π and m_ρ are the experimental numbers, we determine the value of the lattice spacing. This determination, being based on the physical spectrum without reference to any definition of the quark masses, is valid up to $\mathcal{O}(a^2)$. It has been called “lattice-plane method” (ρ - π or K^* - K) in ref. [7]. To determine \tilde{m} and $\tilde{\rho}$, we must study another physical quantity. The best (and most popular) choice is M_{PS}^2 , since it vanishes in the chiral limit and for this reason is very sensitive to the precise value of the quark masses. We fit M_{PS}^2 to

$$a^2 M_{PS}^2 = L_{PS} a (\tilde{m}_1 + \tilde{m}_2) + Q_{PS} a^2 (\tilde{m}_1 + \tilde{m}_2)^2, \quad (3.11)$$

⁵The amputated correlation function of the pseudoscalar density, between external off-shell quark states, is expected to receive in the chiral limit a non-perturbative contribution from the Goldstone pole [16]. We have eliminated such a contribution and determined the renormalization constant Z_P by evaluating the ratio Z_P/Z_S at large values of μ^2 and then calculating Z_P as $(Z_P/Z_S) \cdot Z_S$.

$$a^2 M_{PS}^2 = L'_{PS} a(\tilde{\rho}_1 + \tilde{\rho}_2) + Q'_{PS} a^2(\tilde{\rho}_1 + \tilde{\rho}_2)^2, \quad (3.12)$$

depending whether we use the vector or axial-vector WI. In the above equations, with the values of a fixed from eq. (3.10) and by imposing $M_{PS} = m_\pi$ or m_K , we determine $\tilde{m}_i(\tilde{\rho}_i)$ ⁶. For consistency, quadratic or linear fits should be used both in eq. (3.10) and eq. (3.11) or (3.12). We have also determined the strange quark mass from the vector meson mass, by fitting M_V to

$$aM_V = C_V + L_V a(\tilde{m}_1 + \tilde{m}_2) + Q_V a^2(\tilde{m}_1 + \tilde{m}_2)^2, \quad (3.13)$$

$$aM_V = C_V + L'_V a(\tilde{\rho}_1 + \tilde{\rho}_2) + Q'_V a^2(\tilde{\rho}_1 + \tilde{\rho}_2)^2, \quad (3.14)$$

and using the experimental value of m_ϕ .

From the analysis of the results obtained using quadratic and linear fits, we reached the following conclusions

- As already observed in ref. [8], the coefficient Q_{PS} is large and positive, $Q_{PS} = 1.54(14)$. It is very difficult to explain the sign of Q_{PS} as a consequence of the use of perturbation theory for b_m : such an explanation would require the perturbative value of b_m to be wrong both in size and in *sign*. This is very unlikely (and to our knowledge never happened to be the case). Thus, we believe that the positive curvature Q_{PS} is a physical effect (at least in the quenched approximation).
- In the range of masses used in our calculation, corresponding to the values of $\kappa = 0.1333, 0.1344, 0.1349$ and 0.1352 (from heavier to lighter), the quadratic fits on all the four quarks give consistent results and statistical errors for the quark masses. Moreover, the value of κ_{critic} is in agreement with that obtained from the axial-vector WI.
- With our data, instead, a linear fit of the pseudoscalar mass on all the four values of κ gives a value of κ_{critic} incompatible with that obtained from the axial-vector WI. This confirms that we need a quadratic fit for the pseudoscalar mass.
- Within the accuracy of our results, the coefficient $Q = -2.2 \pm 1.5$ ($Q_V = -2.1 \pm 2.5$) is compatible with zero. Nevertheless, by using the linear or quadratic fit to M_V , the central value of the lattice spacing, and the value of the strange quark mass, vary by about 10%.
- If we assume that the quadratic correction to M_V is negligible, i.e. if we take $Q = 0$ but fit quadratically M_{PS}^2 (as done in ref. [8]), we obtain $a^{-1} = 2.59(12)$ GeV (using $m_\rho - m_\pi$) and $a\tilde{m}_s = 0.031(3)$ in excellent agreement with refs. [6, 8].
- With a quadratic fit for both M_V and M_{PS}^2 , however, we obtain a value of the inverse lattice spacing which is sensibly higher than 2.59 GeV, i.e. $a^{-1} = 2.84(32)$.

⁶ Since we always work with degenerate quarks, we are unable to fit a term $\propto (\tilde{m}_1 - \tilde{m}_2)^2$. In order to separate \tilde{m}_l from \tilde{m}_s (using m_π and m_K), we must, however, distinguish the two quark masses.

Although the two results look compatible within the errors, we believe that the shift is a systematic effect, since the two values of a^{-1} are determined on the same set of configurations. Correspondingly, the values of the quark masses are also modified. In the quadratic case, we get $a\tilde{m}_s = 0.026(5)$. We stress again that the shift of $a\tilde{m}_s$ toward smaller values, although compatible with $0.031(3)$, is a systematic effect.

- It is well known, see for example refs. [2, 3, 7], that the strange quark mass extracted using m_K differs from that obtained from m_ϕ by about 15%. Moreover, the values of the inverse lattice spacing determined from $m_\rho - m_\pi$ or $m_K - m_{K^*}$ are slightly different (by about 100 MeV). These effects were found irrespectively of the action (Wilson, tree-level improved or non-perturbatively improved) used in the numerical calculations and their origin can be traced from the fact that the slope L in eq. (3.10) is smaller than its experimental value. With our data, we find the same effects if we use the linear fit for M_V (with a difference of about 10–15% for $a\tilde{m}_s$, for example). With quadratic fits, we find two nice features: on the one hand the difference in the value of a^{-1} from $m_\rho - m_\pi$ and $m_{K^*} - m_K$ is strongly reduced; on the other, we get about the same strange quark mass from m_K and m_ϕ .

Unfortunately, within our precision, we are unable to fix Q well enough. Nevertheless, at the price of increasing the statistical error, we will take as best estimates of the quark masses those obtained by using quadratic fits for both M_V and M_{PS}^2 . Since this is at present one of the largest sources of uncertainty, we believe that a precise measurement of the quadratic term in the dependence of M_V on M_{PS} is crucial to achieve an accurate determination of the mass of the strange quark.

In table 2, we present a rather extended set of results for the bare and renormalized quark masses, obtained from the vector and axial-vector Ward identities, with the bs computed in boosted perturbation theory. The renormalized masses, in the RI scheme at $\mu = 2$ GeV, have been obtained from the bare ones using the renormalization constants of tab. 1. We see from this table that, by using quadratic fits, denoted as QQ in the table, and non-perturbative Z s, we obtain very consistent results for the two calibrations of the lattice spacing, the masses from the vector or axial-vector WIs and the quark mass extracted from m_K or m_ϕ . In particular, we stress the excellent agreement between the values of the quark masses obtained from the vector and axial-vector WI. This agreement is only possible if one uses Z_S and Z_P computed non-perturbatively. This was first noticed in ref. [7]. A sizeable difference remains, instead, if one uses boosted perturbation theory, see also [6]. For example, with the perturbative values of Z_S and Z_P , we obtain, at $\mu = 2$ GeV, $m_s^{\text{RI}} = 118(12)$ MeV (from \tilde{m}_s) and $m_s^{\text{RI}} = 101(11)$ MeV (from $\tilde{\rho}_s$) instead of 135(14) MeV and 138(16) MeV, respectively.

From the results of the table, we extract our best estimate of the light quark masses in the RI scheme and at $\mu = 2$ GeV (in the following all the results for the running masses refer to $\mu = 2$ GeV)

$$m_l^{\text{RI}} = 5.6(5) \text{ MeV} \quad m_s^{\text{RI}} = 138(15) \text{ MeV} . \quad (3.15)$$

$\beta = 6.2$						
Method	a^{-1} (GeV)	Input	$\tilde{m}_l a$	m_l^{RI} (MeV)	$\tilde{\rho}_l a$	m_l^{RI} (MeV)
$\rho-\pi$ L4Q	2.59(12)	m_π	0.00130(11)	6.1(3)	0.00127(11)	6.0(3)
K^*-K L4Q	2.69(12)	m_π	0.00120(10)	5.9(3)	0.00117(10)	5.8(3)
$\rho-\pi$ QQ	2.84(32)	m_π	0.00108(23)	5.6(6)	0.00105(24)	5.5(6)
K^*-K QQ	2.83(25)	m_π	0.00109(18)	5.6(5)	0.00106(18)	5.5(5)
Method	a^{-1} (GeV)	Input	$\tilde{m}_s a$	m_s^{RI} (MeV)	$\tilde{\rho}_s a$	m_s^{RI} (MeV)
$\rho-\pi$ L4Q	2.59(12)	m_K	0.0312(26)	147(6)	0.0313(28)	149(7)
		m_ϕ	0.0363(35)	171(9)	0.0379(40)	181(12)
K^*-K L4Q	2.69(12)	m_K	0.0290(23)	142(6)	0.0291(24)	144(6)
		m_ϕ	0.0319(24)	156(5)	0.0331(27)	164(7)
$\rho-\pi$ QQ	2.84(32)	m_K	0.0261(55)	135(14)	0.0261(58)	137(16)
		m_ϕ	0.0260(84)	134(29)	0.0265(94)	139(35)
K^*-K QQ	2.83(25)	m_K	0.0263(42)	135(11)	0.0263(45)	137(12)
		m_ϕ	0.0264(58)	136(19)	0.0270(66)	141(23)

Table 2: Improved bare masses of the light ($\tilde{m}_l a = (\tilde{m}_u a + \tilde{m}_d a)/2$ and $\tilde{\rho}_l a = (\tilde{\rho}_u a + \tilde{\rho}_d a)/2$) and strange ($\tilde{m}_s a$ and $\tilde{\rho}_s a$) quarks in lattice units. The improvement coefficients b_m , b_A and b_P necessary to obtain \tilde{m}_l , $\tilde{\rho}_l$, etc., have been taken from boosted perturbation theory. The renormalized quark masses in the RI scheme at a renormalization scale $\mu = 2$ GeV, obtained using the calibration of the lattice spacing given in this table, and the non-perturbative values of the Z s from tab. 1, are also given. We use the values of the inverse lattice spacing obtained from $m_\rho - m_\pi$ (denoted as $\rho - \pi$) and from $m_{K^*} - m_K$ (denoted as $K^* - K$). L4Q denotes the same fitting procedure as in ref. [8]: a linear fit for M_V , over the four quark masses, combined with a quadratic fit for M_{PS}^2 . These results can be directly compared to those of refs. [8]. QQ indicates that quadratic fits were used for both M_V and M_{PS}^2 .

Note that, to obtain the results in eq. (3.15), we never used perturbation theory but for the values of b_m , b_A and b_P . However, as discussed above, this is expected to be a source of negligible uncertainty in the determination of light quark masses.

4 \overline{MS} and RGI masses

Perturbation theory only enters if we want to convert the results of the RI scheme into the \overline{MS} scheme, which has been adopted as the standard one in the literature (both in lattice and QCD sum rule calculations). Although this is not necessary (and we believe that m^{RGI} is a more convenient definition, see below), for comparison with other determinations, we also give the quark masses in \overline{MS} . These are found using the relation

$$m^{\overline{MS}}(\mu) = R_m m^{RI}(\mu) , \quad (4.16)$$

where, following the notation of ref. [29]

$$R_m = 1 + \frac{\alpha_s(\mu)}{(4\pi)} (Z_m^{RI})_0^{(1)} + \frac{\alpha_s^2(\mu)}{(4\pi)^2} (Z_m^{RI})_0^{(2)} + \dots \quad (4.17)$$

with

$$\begin{aligned} (Z_m^{RI})_0^{(1)} &= -8 \frac{(N_c^2 - 1)}{4 N_c} , \\ (Z_m^{RI})_0^{(2)} &= \frac{(N_c^2 - 1)}{96 N_c^2} \left(-75 - 2645 N_c^2 + 288 \zeta_3 + 576 N_c^2 \zeta_3 + 356 N_c n_f \right) \end{aligned} \quad (4.18)$$

In the above equation, $\zeta_3 = 1.20206\dots$ is the Riemann zeta function. Since all previous results have been obtained at the NLO, i.e. by ignoring the corrections due to $(Z_m^{RI})_0^{(2)}$, we give the results both at the NLO and at the N²LO order. In the numerical evaluation of R_m we have used $n_f = 4$. The reason for this choice, in spite of the quenched approximation adopted in our simulation, is the following. The mass $m^{RI}(\mu)$ is to be interpreted as the mass in the continuum which includes a systematic (unknown) error coming from the quenched approximation. Thus, $m^{RI}(\mu)$ is the estimate of the physical value of the quark mass at the scale $\mu = 2$ GeV, at which $n_f = 4$. By fixing in both cases $\alpha_s(M_Z) = 0.118$, which corresponds to $\alpha_s^{NLO}(\mu = 2 \text{ GeV}) = 0.296$ and $\alpha_s^{N^2LO}(\mu = 2 \text{ GeV}) = 0.300$, we obtain $R_m^{NLO} = 0.874$ and $R_m^{N^2LO} = 0.804$ which give

$$\begin{aligned} m_l^{NLO \overline{MS}} &= 4.9(4) \text{ MeV} , \quad m_s^{NLO \overline{MS}} = 121(13) \text{ MeV} \\ m_l^{N^2LO \overline{MS}} &= 4.5(4) \text{ MeV} , \quad m_s^{N^2LO \overline{MS}} = 111(12) \text{ MeV} . \end{aligned} \quad (4.19)$$

Note that *by using BPT*, we would have been obtained, for example, $m_l^{NLO \overline{MS}} = 4.3$ MeV and $m_s^{NLO \overline{MS}} = 98(11)$ MeV (from the vector WI, which is the most favorable case for BPT).

We now compare our results to those of refs. [6] and [8]. For $\mu = 2$ GeV, we rescale their results with the ratios of the perturbative Z s to the non-perturbative ones, at the corresponding values of β . In this way, using the results of ref. [6], we get $m_s^{NLO \overline{MS}} = 126(5)$ MeV from the vector WI and $m_s^{NLO \overline{MS}} = 138(3)$ MeV from the axial-vector WI and, from ref. [8], $m_s^{NLO \overline{MS}} = 127(15)$ MeV from the vector WI. These numbers are in good agreement with the NLO result given in eq. (4.19).

We also analyzed the data of our test run at $\beta = 6.0$. In this case we obtain rather larger values: $m_s^{NLO \overline{MS}} = 145(7)$ MeV from the vector WI, using quadratic fits ($m_s^{NLO \overline{MS}} = 145(14)$ MeV from the axial WI). With linear fits on the lightest quarks, corresponding to $\kappa = 0.1324, 0.1333$ and 0.1342 , we obtain a result much closer to that obtained at $\beta = 6.2$, i.e. $m_s^{NLO \overline{MS}} = 126(6)$ MeV. As a further check, we have fitted the results of ref. [6] with our programs (with quadratic fits), obtaining $m_s^{NLO \overline{MS}} = 131$ MeV⁷. The problem is that our accuracy at $\beta = 6.0$ is rather poor, and we attribute the discrepancy between the results with quadratic fits and all the others to the fact that we do not control, in this case, the quadratic corrections. These are affected by some differences with the more precise results of ref [6], which we have found for the values of the meson masses for the lightest quarks.

A originally discussed in ref. [30] for lattice calculations, and more recently stressed by the Alpha collaboration, a more convenient definition of the quark mass is m^{RGI} which, analogously to the RGI B -parameter, \hat{B}_K , is also renormalization scheme and scale invariant. Moreover, m^{RGI} can be directly related to the mass parameters appearing in the fundamental Lagrangian of grand unified theories, at the unification scale. The relation between m^{RGI} and $m^{RI}(\mu)$ is given by [29]

$$m_{n_f=4}^{RGI} \equiv R_{RGI} m^{RI}(\mu) = \alpha_s(\mu)^{-12/25} \left[1 - \frac{\alpha_s(\mu)}{4\pi} \left(\frac{17606}{1875} \right) \right. \quad (4.20)$$

$$\left. - \frac{\alpha_s^2(\mu)}{(4\pi)^2} \left(\frac{3819632767}{21093750} - \frac{952}{15} \zeta_3 \right) \right] m^{RI}(\mu) . \quad (4.21)$$

With the same values of α_s given above, we find $R_{RGI}^{NLO} = 1.40$ and $R_{RGI}^{N^2LO} = 1.28$, corresponding to

$$\begin{aligned} \hat{m}_l^{NLO} &= 7.8(7) \text{ MeV} , & \hat{m}_s^{NLO} &= 193(21) \text{ MeV} \\ \hat{m}_l^{N^2LO} &= 7.2(6) \text{ MeV} , & \hat{m}_s^{N^2LO} &= 177(19) \text{ MeV} . \end{aligned} \quad (4.22)$$

5 Conclusions

Using the $\mathcal{O}(a^2)$ improved Alpha action, we have computed the light-quark masses renormalized non-perturbatively. The NPM used in this study removes the discrepancies, observed in previous calculations, between the results obtained using the vector and the

⁷ It is not possible to extract an error in this case.

axial-vector Ward identities. It also gives values of the quark masses larger than those obtained by computing the renormalization constants with (boosted) perturbation theory. We use perturbation theory only to translate our results either to the \overline{MS} scheme, which has been adopted in most of the determinations of the quark masses, or to our preferred definition which is the RGI mass. Using the results of ref. [29] this can be done at N^2LO accuracy.

A further reduction of the systematic errors would require the non-perturbative determination of the bs (although we believe that this uncertainty is, for light quarks, by far smaller than the others); the evaluation of $Z_S(\mu)$ and $Z_P(\mu)$ at larger physical scales, μ , and smaller values of μa , a fit of the renormalized masses in a^2 (in order to remove the residual discretization errors) and, of course, precise unquenched calculations. We also found that a precise determination of the quadratic dependence of the vector meson mass on the quark mass is important to understand the differences in \tilde{m}_s determined using different mesons (~ 20 MeV).

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